

Anomalous BRST Complexes for Non-Critical Massive Strings.

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Abstract

It is shown that the BRST resolution of the spaces of physical states of non-critical (anomalous) massive string models can be consistently defined. The appropriate anomalous complexes are obtained by canonical restrictions of the ghost extended spaces to the kernel of curvature operator and additional Gupta-Bleuler like conditions without any modifications of the matter sector. The cohomologies of the polarized anomalous complex are calculated and analyzed in details.

Introduction

The massive string model was introduced in its full extent in the paper [1] as a step towards the consistent and effective description of the strings (treated as 1-dimensional extended objects) in sub-critical dimensions $1 < d < 26$. The classical model was defined by the variational principle with standard Nambu-Goto action functional (in its BDHP form [2]) supplemented by the term to govern the motion of an additional scalar - the Liouville field.

On the level of canonical formulation, the massive string model is the constrained system of mixed type. Almost all constraints (except of the kinetic one) are of second class.

The standard quantization procedure applied to the classical massive string results in relativistic, constrained quantum system with anomaly. The anomaly contains both classical and quantum contributions. The presence of the anomaly enforces the application of the Gupta-Bleuler like procedure [3] (polarization of constraints) to define the subspace of physical degrees of freedom (10).

By the use of the general results on the structure of the Verma modules of the Virasoro algebra [4] it was proved that the family of unitary (and formally also relativistically invariant) quantum massive string models can be distinguished. This family decays into two series: continuous (11) and discrete one (12).

The particle content of these series was analyzed and completely determined in [5]. The form of their spectra makes them interesting and promising from the point of view of the applications to the description of the interactions of low-energy (composite) QCD states.

The interactions of one-dimensional extended objects appeared to be most efficiently described within the framework of cohomological BRST formalism [6]. On the other hand it was shown [7] that only critical massive strings (13) located at the very beginning of the discrete series do admit the BRST resolution in its standard form. The reason is that only for these critical models the curvature operator (the square of differential) does vanish identically - the ghost-extended system is anomaly free.

This paper is devoted to the natural generalization of the canonical BRST approach, which enables one to look at all models from the universal point of view. The formalism can be applied to the whole class of unitary strings¹ containing both: critical (anomaly free) and non-critical (anomalus) models. The proposed approach, in contrast to the schemes applied earlier [8],[9] does not introduce any additional degrees of freedom - neither in matter nor in ghost sector. The main idea consists in restricting the appropriate differential space.

There are two complexes which one may associate with anomalus models. The first one, called the anomalus complex, is defined as the subspace annihilated by the anomaly (curvature) operator - the square of the canonically constructed BRST differential. The cohomologies of this complex are non-vanishing at higher ghost numbers and are quite complicated to be completely determined.

The second complex, called the polarized anomalus complex, provides the cohomological resolutions of the spaces of physical states for all unitary models. This means that

¹From the formal point of view it can be applied to all non-unitary models too.

the critical model can be thought of as the limiting case also from the point of view of BRST formulation - in this sense that the underlying space of the corresponding complex is the largest possible.

It is worth to make one comment on the problem of null vectors in the space of physical states (10) in the context of BRST formulation. In the critical case all the null states (radical) are cohomologically trivial and can be eliminated by BRST-gauge transformations [7]. The corresponding No-Ghost Theorem can be proved by very general cohomological methods [6],[10].

The results presented in this paper indicate that the relation between the presence of null states and (cohomological) gauge symmetries can be directly established in the critical case only. Indeed, the Theorem 3.1 clearly states that there are non-trivial relative cohomology classes, which according to e.g. [11] are null. A better understanding of this appearance needs further investigation.

The considerations of this paper are restricted to massive string models for at least two reasons. On the one hand side the anomalies present in these models are of mixed nature: there are classical and quantum contributions. On the other hand their structure is simple enough to perform the detailed analysis of polarized anomalous cohomologies. It is however evident that the proposed approach can be applied to other constrained systems too. It should give the analogous results, namely cohomological resolutions of the corresponding spaces of physical states by the polarized complex. It should also be mentioned that the property that the elements of the anomalous complex are the highest weight vectors of $sl(2, \mathbb{C})$ algebra (41),(42) generated by the anomaly operator seems to be universal within the wide class of constrained systems.

The paper is organized as follows.

In the first section, for the sake of completeness, the sketch of massive string model is briefly presented. The No-Ghost Theorem is recalled in order to restrict further considerations to unitary non-critical massive string models.

In the second chapter the anomalous complexes for string models are constructed: they are defined as subspaces of standard ghost extended string spaces (called string differential spaces throughout this paper). It appears that anomalous complexes inherit the essential properties of critical massive string BRST complex, in particular the bigraded structure of the relative cochains.

The next chapter is devoted to the computation and analysis of the cohomologies of the polarized anomalous BRST complex. The bigraded cohomologies (Dolbeaut-Grotendieck) as well as relative cohomologies are calculated and represented in terms of physical string states of "old covariant formalism". It is thus shown, that also in non-critical case, the polarized anomalous BRST complex provides the resolution of the space of physical string states. The absolute cohomologies of the polarized complex are also reconstructed.

Finally, the concluding remarks are added, and some open question are raised.

1 The classical and quantum massive strings

A brief description of the classical and quantum massive string models [1] is presented in this section.

1.1 The classical massive string

The classical massive string model is defined by the functional [1]:

$$S[M, g, \varphi, x] = - \frac{\alpha}{2\pi} \int_M \sqrt{-g} d^2z g^{ab} \partial_a x^\mu \partial_b x^\nu - \frac{\beta}{2\pi} \int_M \sqrt{-g} d^2z (g^{ab} \partial_a \varphi \partial_b \varphi + 2R_g \varphi) , \quad (1)$$

which is an extension of the standard d -dimensional string world-sheet action by an additional term, which governs the motion of the scalar $\varphi(z)$ called the Liouville field. The detailed analysis [1] of the variational problem for (1) leads to the constrained phase space system. The phase space of this system is parametrized by real canonical pairs (x^μ, p_μ) with

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \quad \mu, \nu = 0, \dots, d-1, \quad (2)$$

to describe the center-of-mass motion, and the complex variables a_m^μ, u_m ($m \in \mathbb{Z} \setminus \{0\}$) related to the oscillatory degrees of freedom. The variables a_m^μ, u_m are constructed out of higher Fourier modes of the real fields $x^\mu(z)$ and $\varphi(z)$ and their canonically conjugated momenta. Their Poisson brackets are of standard form:

$$\{a_m^\mu, a_n^\nu\} = im\eta^{\mu\nu} \delta_{m+n}, \quad \{u_m, u_n\} = im\delta_{m+n}, \quad \mu, \nu = 0, \dots, d-1. \quad (3)$$

The classical constraints of the system correspond to, roughly speaking, the invariance of (1) with respect to the diffeomorphisms of the world-sheet. They are given by the generators of hamiltonian action of the gauge group on the phase space. The complex modes of the canonical generators of this action are given by the following expressions:

$$l_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_{m+k} \cdot a_{-k} + \frac{1}{2} \sum_{k \in \mathbb{Z}} u_{m+k} u_{-k} + 2i\sqrt{\beta} m u_m, \quad m \in \mathbb{Z}. \quad (4)$$

The Poisson bracket algebra of the above modes does not close. It is modified by non-zero central term - the Lie algebra scalar cocycle:

$$\{l_m, l_n\} = i(m-n)l_{m+n} - 4i\beta m^3 \delta_{m+n}. \quad (5)$$

The presence of the central term in the Poisson bracket relations (5) indicates that the constraints are of mixed type with l_0 being the unique constraint of first class. The corresponding gauge group action is called [12] weakly hamiltonian in this case.

1.2 The quantum massive string - the states in "old covariant quantization"

Basic definitions

The space of states of massive string model is most conveniently defined as a direct integral:

$$\mathcal{H} = \int d^d p \mathcal{H}(p) , \quad (6)$$

of pseudo-unitary Fock spaces. Every $\mathcal{H}(p)$ is generated by the algebra of excitation operators:

$$\begin{aligned} [a_m^\mu, a_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n} , \\ [u_m, u_n] &= m\delta_{m+n} \quad m, n \in \mathbb{Z} \quad \mu, \nu = 0, \dots, d-1 . \end{aligned} \quad (7)$$

out of the vacuum vector $\omega(p)$ satisfying $a_m^\mu \omega(p) = 0 = u_m \omega(p)$; $m > 0$. The vacuum vectors are generalized eigenfunctions for the momentum operators: $P^\mu \omega(p) = p^\mu \omega(p)$ and are formally normalized by the condition: $(\omega(p), \omega(p')) = \delta(p - p')$.

The unique scalar product in (6) is defined by imposing formal conjugation rules $a_m^{\mu*} = a_{-m}^\mu$, $u_m^* = u_{-m}$ on string modes.

The operators corresponding to the constraints are obtained by replacing the classical modes (3) by their operator counterparts in the expressions (4) and normal ordering:

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{m+k} \cdot a_{-k} : + \frac{1}{2} \sum_{k \in \mathbb{Z}} : u_{m+k} u_{-k} : + 2i\sqrt{\beta} m u_m + 2\beta \delta_{m0} . \quad (8)$$

The classical central term in (5) gets modified by normal ordering anomaly:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12} m (m^2 - 1) (d + 1 + 48\beta) \delta_{m+n} . \quad (9)$$

The anomaly in the quantum constraints algebra has two sources. There is trace of the classical central term and strictly quantum contribution.

The space of physical states and No-Ghost Theorem

Within the framework of "old covariant" quantization procedure, the space of physical states is defined as the subspace of (6) consisting of vectors subject to the infinite set of equations:

$$\mathcal{H}_{\text{phys}} = \{ \Psi ; (L_m - a\delta_{m0}) \Psi = 0 , m \geq 0 \} . \quad (10)$$

The real parameter a defines the beginning of the string mass spectrum and is left to be fixed by natural consistency conditions: the unitarity and relativistic invariance. The unitarity condition means that the inner product in the space (10) should be non-negative. Then the quotient of (10) by the subspace of all null vectors (radical) gives (after completion) the Hilbert space of states.

According to the No-Ghost theorem [1] one can distinguish two principal families of unitary string models.

There is continuous series:

$$a \leq 1 \quad , \quad 0 < \beta \leq \frac{24 - d}{48} , \quad (11)$$

and discrete one:

$$\begin{aligned}\beta &= \frac{24-d}{48} + \frac{1}{8m(m+1)} ; \quad m \geq 2 \\ a &= 1 - \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} ; \quad 1 \leq r \leq m-1, \quad 1 \leq s \leq r .\end{aligned}\quad (12)$$

The beginning of the discrete series ($m = 2$) corresponds to so called critical massive string model with:

$$\beta = \frac{25-d}{48} \quad \text{and} \quad a = 1 . \quad (13)$$

It was shown in [7] that in this case the space of physical states admits standard BRST resolution. It was also shown that in the case of (13) all the elements of the radical correspond to pure gauge degrees of freedom in the sense of BRST cohomology.

The construction of appropriate BRST-like complexes for massive string models corresponding to non-critical (anomalous) values of the parameters (11,12) will be given in the next section.

2 Anomalous BRST complexes

For the reader's convenience the section is divided into three parts containing ingredients and subsequent steps of the construction. The first two steps are common to both: critical and anomalous models and may be found in the literature (e.g. [13]). They are repeated here for the sake of completeness and in order to stress that one is able to split the differential spaces and corresponding complexes into finite dimensional subspaces and subcomplexes. This splitting plays an essential role in the computation of cohomologies.

2.1 The canonical ghost sector

In order to construct the anomalous complexes one starts with the same ghost differential space² as in the case of critical massive string models [7]. Despite of the presence of the anomaly (9) the quantum constraints (8) are treated as if they were of first class.

With every constraint L_m , $m \in \mathbb{Z}$ (8) one associates a pair of fermionic operators: the ghost c_m and the anti-ghost b_m . It is assumed that they satisfy the following anti-commutation relations:

$$\{b_m, c_n\} = \delta_{m+n} , \quad \{b_m, b_n\} = 0 , \quad \{c_m, c_n\} = 0 , \quad m, n \in \mathbb{Z} . \quad (14)$$

The ghost sector \mathcal{C}_{gh} is created out of the vacuum of the ghost space, which is formally defined as a vector ω satisfying:

$$c_m \omega = 0 ; \quad m > 0 \quad \text{and} \quad b_m \omega = 0 ; \quad m \geq 0 . \quad (15)$$

²Differential space is by definition a pair $(\mathcal{C}, \mathcal{D})$ consisting of graded space \mathcal{C} and differential \mathcal{D} of order ± 1 , which is not necessarily nilpotent.

The space \mathcal{C}_{gh} is equipped with the (non-degenerate) scalar product by imposing the following normalization condition: $(\omega, c_0 \omega) = 1$ of the ghost vacuum, and the following formal conjugation rules:

$$b_m^* = b_{-m} \quad , \quad c_m^* = c_{-m} \quad , \quad (16)$$

of the ghost modes.

The differential on the space \mathcal{C}_{gh} is constructed in most convenient way [13] with the help of the natural realization of the constraint algebra on the ghost space:

$$\mathcal{L}_m = \sum_{k \in \mathbb{Z}} (k - m) : c_{-k} b_{m+k} : \quad . \quad (17)$$

The normal ordering prescription in the formulae above is determined, as usually, by the conditions imposed on the vacuum (15) and is supplemented with the antisymmetrization rule for the ghost zero-modes : $c_0 b_0 := \frac{1}{2}(c_0 b_0 - b_0 c_0)$. The resulting operators \mathcal{L}_m , $m \in \mathbb{Z}$ satisfy the following Virasoro algebra relations:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{m+n} - \delta_{m+n} \left(\frac{26}{12} (m^3 - m) + 2m \right) \quad . \quad (18)$$

The differential in \mathcal{C}_{gh} is defined to be:

$$d_{\text{gh}} = \frac{1}{2} \sum_{m > 0} c_{-m} \mathcal{L}_m + \frac{1}{2} c_0 \mathcal{L}_0 + \frac{1}{2} \sum_{m > 0} \mathcal{L}_{-m} c_m \quad , \quad (19)$$

and because of the anomaly in (18) it is not nilpotent:

$$d_{\text{gh}}^2 = - \sum_{m > 0} \left(2m + \frac{26}{12} m(m^2 - 1) \right) c_{-m} c_m \quad . \quad (20)$$

One can see that the ghost level operator $\mathcal{L}_0 = \sum m : c_{-m} b_m :$ is diagonalizable in \mathcal{C}_{gh} and it commutes (in contrast to remaining operators of (17)) with the differential (19). Therefore the space \mathcal{C}_{gh} splits into a direct sum of finite dimensional subspaces of fixed ghost level N which are d_{gh} invariant:

$$\mathcal{C}_{\text{gh}} = \bigoplus_{N \geq 0} \mathcal{C}_{\text{gh}}^N \quad . \quad (21)$$

Every subspace $\mathcal{C}_{\text{gh}}^N$ of fixed level is graded:

$$\mathcal{C}_{\text{gh}}^N = \bigoplus_r \mathcal{C}_{\text{gh}}^{N,r} \quad , \quad d_{\text{gh}} : \mathcal{C}_{\text{gh}}^{N,r} \rightarrow \mathcal{C}_{\text{gh}}^{N,r+1} \quad , \quad (22)$$

by half integral eigenvalues of the ghost number operator:

$$\text{gh} = \sum_{m \in \mathbb{Z}} : c_{-m} b_m : = \sum_{m > 0} (b_{-m} c_m - c_{-m} b_m) + \frac{1}{2} (c_0 b_0 - b_0 c_0) \quad . \quad (23)$$

The direct sum in the ghost number decomposition (22) ranges over finite and symmetric with respect to zero interval of half-integers. The spaces $\mathcal{C}_{\text{gh}}^{N,r}$ and $\mathcal{C}_{\text{gh}}^{N,-r}$ are paired in a non-degenerate way with respect to the scalar product defined by the vacuum normalization condition and formal conjugation rules (16).

2.2 The string differential spaces

The string differential space is constructed as an appropriate tensor product of the covariant string space \mathcal{H} (6) and the ghost sector \mathcal{C}_{gh} described in the previous subsection. More precisely:

$$\mathcal{C} = \bigoplus_{N \geq 0} \int d^d p \mathcal{C}^N(p) \quad \text{where} \quad \mathcal{C}^N(p) = \bigoplus_{N' + N'' = N} \mathcal{H}^{N'}(p) \otimes \mathcal{C}_{\text{gh}}^{N''} . \quad (24)$$

The decompositions of the spaces $\mathcal{H}(p)$ of (6):

$$\mathcal{H}(p) = \bigoplus_{N' \geq 0} \mathcal{H}^{N'}(p) . \quad (25)$$

into finite dimensional eigensubspaces of string level operator $R_{\text{str}} = L_0 - \frac{1}{2\alpha} p^2 - 2\beta$ were used in the definition (24).

The string differential space is equipped with the grading inherited from the ghost sector (22). The subspaces of $\mathcal{C}^N(p)$ of opposite ghost number are mutually dual with respect to the canonical pairing induced on the tensor product (24).

The differential in this space is defined in the standard way [13]:

$$D = \sum_{m \in \mathbb{Z}} (L_m - \delta_{m0} a) \otimes c_{-m} + 1 \otimes d_{\text{gh}} . \quad (26)$$

Introducing the following operator:

$$J_{\xi}^{(+)} = \sum_{m > 0} r_{\xi}(m) c_{-m} c_m , \quad (27)$$

where:

$$r_{\xi}(m) = m(m^2 - \xi) , \quad \xi = 1 + \frac{24(1 - a)}{(d - 25) + 48\beta} , \quad (28)$$

one may check that the square of the differential equals to:

$$D^2 = \frac{1}{12} ((d - 25) + 48\beta) J_{\xi}^{(+)} , \quad (29)$$

i.e. the operator (26) is not nilpotent in general.

Lack of the nilpotency (29) implies that the differential is also not invariant with respect to the ghost extended constraint operators:

$$L_m^{\text{tot}} = \{ b_m, D \} = L_m + \mathcal{L}_m - \delta_{m0} a , \quad m \in \mathbb{Z} . \quad (30)$$

From the general identity $[L_m^{\text{tot}}, D] = [b_m, D^2]$ it immediately follows that:

$$[L_m^{\text{tot}}, D] = \frac{1}{12} ((d - 25) + 48\beta) r_{\xi}(m) c_m . \quad (31)$$

Therefore the differential space (\mathcal{C}, D) is not the complex except of the case when the parameters β and a take their critical values (13). In this very special case the differential space constructed above is equipped with nilpotent differential operator and is used to be called the (standard) BRST-complex.

This standard structure was introduced in [14] for $d = 26$ Nambu-Goto critical string model and in [7] for critical massive string models with Liouville modes. In this last paper the cohomologies of critical BRST-complexes were calculated and described in details.

In the remaining part of this chapter the complexes corresponding to anomalous square in (29) and to generic cases of unitary non-critical massive string models (11) and (12) will be introduced and analyzed.

2.3 Anomalous complexes

The constructions of this subsection can be applied to any physical systems with anomalies. For reasons explained in the Introduction all further considerations will be concentrated on string models however.

The space of the anomalous complex is defined in the simplest possible way, namely as the the maximal subspace of \mathcal{C} (24) on which the canonical differential (26) is nilpotent. It is clear that it coincides with the kernel of the anomaly operator (29). The space of the polarized anomalous complex is defined as the maximal subspace of \mathcal{C} on which the differential (26) is invariant (see (31)) with respect to Gupta-Bleuler subalgebra (10) of that formed by all ghost extended constraint operators (30). The remarks above are formalized in the following

Definition 2.1 (Anomalous complexes)

1. *A differential space*

$$(\mathcal{A}, D) \text{ where } \mathcal{A} := \{ \Psi \in \mathcal{C} ; D^2 \Psi = 0 \} = \ker D^2 \quad (32)$$

is to be said anomalous BRST complex.

2. *A differential space*

$$(\mathcal{P}, D) \text{ where } \mathcal{P} := \{ \Psi \in \mathcal{C} ; [L_m^{\text{tot}}, D] \Psi = 0 ; m \geq 0 \} \quad (33)$$

is to be said polarized anomalous BRST complex.

The definition (32) of the anomalous complex is by all means consistent from the formal point of view: the kernel of D^2 is preserved by the action of D .

The space (33) of polarized complex needs more care. From the formulae (31) one immediately obtains $D^2 = \sum c_{-m} [L_m^{\text{tot}}, D]$. Consequently the space \mathcal{P} is contained in \mathcal{A} and $D^2 \mathcal{P} = 0$. Hence the necessary condition for (33) to be a complex is satisfied. It is however not obvious that the differential D acts inside \mathcal{P} . It depends on the properties of the anomaly which are encoded in the coefficients $r_\xi(m)$ of the operator $J_\xi^{(+)}$ of (27).

In order to decide if $D\mathcal{P} \subset \mathcal{P}$ one should first notice that, in any case, the space \mathcal{P} is preserved by the action of the ghost annihilation operators:

$$\text{if } \Psi \in \mathcal{P} \text{ then } b_m \Psi \in \mathcal{P}, \quad m \geq 0. \quad (34)$$

The above property follows from the general identity $\{b_k, [L_m^{\text{tot}}, D]\} = (m - k)L_{m+k}^{\text{tot}} + [L_k^{\text{tot}}, L_m^{\text{tot}}]$ and the fact that the subalgebra formed by $L_m^{\text{tot}}, m \geq 0$ is anomaly free.

Next, admitting (for the moment) arbitrary real values of the parameter ξ in the definitions (27) and (28) one may distinguish two (essentially) different cases. The first one corresponds to the non-degenerate anomaly and is defined by the following conditions on the parameter ξ : $r_\xi(m) \neq 0$ for all $m > 0$. In the second, opposite case, $\xi = s^2$ for some integer $s > 0$ and the anomaly becomes degenerate at the s -th ghost mode: $r_\xi(s) = 0$.

One is now in a position to prove the following

Lemma 2.1

The space (\mathcal{P}, D) is a complex iff either the anomaly is non-degenerate or it is degenerate at the 1-st ghost mode.

Proof: According to what was said above it is enough to prove that the space \mathcal{P} is preserved by the differential D .

In the non-degenerate case the conditions of the definition (33) amount to $c_m \Psi = 0$ for all $m > 0$. Hence \mathcal{P} is generated by all ghost modes. The differentials of the c_m ghost modes can be explicitly calculated ($m > 0$):

$$\{D, c_m\} = - \sum_{k>0} (m+2k) c_{-k} c_{m+k} + \frac{1}{2} \sum_{k=1}^{m-1} (m-2k) c_{m-k} c_k - m c_0 c_m . \quad (35)$$

The formulae above clearly implies that $\{D, c_m\} |_{\mathcal{P}=0}$ and $D\mathcal{P} \subset \mathcal{P}$ in non-degenerate case.

Assume now that the anomaly is degenerate at s -th mode: $r_\xi(s) = 0$. Then the elements of \mathcal{P} are defined by the conditions $c_m \Psi = 0$ for all $m > 0$ except of $m = s$. In this case the space \mathcal{P} is generated by all ghost modes and one anti-ghost b_{-s} . Hence the state $\Psi = b_{-s} c_0 \Omega(p)$, where $\Omega(p)$ is the ghost-matter vacuum located at some momentum p , belongs to \mathcal{P} . Assuming that $D\Psi \in \mathcal{P}$ one obtains $L_{-s}^{\text{tot}} c_0 \Omega(p) \in \mathcal{P}$ as $Dc_0 \Omega(p) = 0$. Due to (34) one gets $b_1 L_{-s}^{\text{tot}} c_0 \Omega(p) = (s+1) b_{-s+1} c_0 \Omega(p) \in \mathcal{P}$, which for $s > 1$ yields contradiction. One is left with one possibility only, namely: $s = 1$. From the formulae (35) one may easily see that for $m > 1$ the differential $\{D, c_m\}$ does never contain c_1 ghost mode, which is not multiplied by the higher one. Therefore $\{D, c_m\} |_{\mathcal{P}=0}$ in this unique degenerate case. \square

It is worth to stress that for both series (11) and (12) of unitary massive string models one has $\xi \leq 1$. Therefore the polarized complexes always exist and moreover the only admissible degenerate case occurs in the continuous series for $a = 1$.

It has to be also mentioned that the spaces introduced in the definitions (32) and (33) depend on the parameter ξ (28) of the string models. For the sake of simplicity this dependence will never be marked in the notation explicitly.

In order to say more on the structure of the anomalous complex it should be noted that it can be decomposed in the same way as total string differential space (24):

$$\mathcal{A} = \bigoplus_{N \geq 0} \int d^d p \mathcal{A}^N(p) , \quad (36)$$

i.e. into a direct sum/integral of subcomplexes supported by the subspaces of fixed momentum and level. The polarized complex admits an analogous decomposition.

Due to the fact that the anomaly (29) acts in the matter sector as multiplication by (non-zero) number, the fixed level and momentum components in (36) can be factorized in the same way as in full differential space (24):

$$\mathcal{A}^N(p) = \bigoplus_{N'+N''=N} \mathcal{H}^{N'}(p) \otimes \mathcal{A}_{\text{gh}}^{N''} , \quad (37)$$

but with restricted ghost factors this time:

$$\mathcal{A}_{\text{gh}}^{N''} = \{ c \in \mathcal{C}_{\text{gh}}^{N''} ; J_{\xi}^{(+)} c = 0 \} = \ker J_{\xi}^{(+)} . \quad (38)$$

Since the ghost number operator (23) preserves the kernel of D^2 : $[\text{gh}, D^2] = 2D^2$ the spaces $\mathcal{A}^N(p)$ admit the grading induced from (22) i.e. every element in the kernel of $J_{\xi}^{(+)}$ can be decomposed into components of fixed degree. Hence:

$$\mathcal{A}^N(p) = \bigoplus_r \mathcal{A}^{N,r}(p) \quad \text{and} \quad D : \mathcal{A}^{N,r}(p) \rightarrow \mathcal{A}^{N,r+1}(p) . \quad (39)$$

Not all the properties of the original grading (22) are inherited from the total ghost sector. It will be made evident that all ghost numbers lower than $-\frac{1}{2}$ for non-degenerate anomaly and below $-\frac{3}{2}$ in degenerate case are excluded by (38). For this reason the canonical pairing induced from the total complex becomes degenerate on \mathcal{A} and \mathcal{P} .

Depending on the case one introduces the complementary operator:

$$J_{\xi}^{(-)} = \begin{cases} \sum_{m>0} \frac{1}{r_{\xi}(m)} b_{-m} b_m & ; \quad \xi < 1 \\ \sum_{m>1} \frac{1}{r_{\xi}(m)} b_{-m} b_m & ; \quad \xi = 1 . \end{cases} \quad (40)$$

The operator $J_{\xi}^{(-)}$ neither commutes with D nor it does with D^2 . Therefore $J_{\xi}^{(-)}$ does not preserve the space \mathcal{A} underlying the anomalous complex (32). Nevertheless it is very useful in closer identification of the structure of the subspaces $\mathcal{A}^N(p)$ of the anomalous complex.

The commutator of $J_{\xi}^{(-)}$ with the anomaly operator $J_{\xi}^{(+)}$ is almost ξ independent, more precisely:

$$[J_{\xi}^{(+)}, J_{\xi}^{(-)}] = J^{(0)} := \text{gh} - \text{gh}_{(0)} - \delta_{(\xi-1)} \text{gh}_{(1)} , \quad (41)$$

where gh is the ghost number operator of (23) and $\text{gh}_{(m)}$ denotes its m -th mode component.

From the commutation relations:

$$[J^{(0)}, J_{\xi}^{(\pm)}] = \pm 2J_{\xi}^{(\pm)} , \quad (42)$$

it follows that $J_{\xi}^{(\pm)}$ together with $J^{(0)}$ form the structure of $sl(2, \mathbb{C})$ Lie algebra.

Hence, the subspaces $\mathcal{A}^N(p)$ of the anomalous complex consist of all elements of $\mathcal{C}^N(p)$, which are of highest weights (coprimitive cochains [15]) with respect to the above $sl(2, \mathbb{C})$ -Lie algebra.

It is well known (e.g. [16]) that the commutation relations (41) imply that the highest weight vectors have non-negative weights with respect to $J^{(0)}$. The operator $J^{(0)}$ does not feel the ghost zero-modes in non-degenerate case and in addition c_{-1} , b_{-1} ghost modes for $\xi = 1$.

From the relation (42) it follows that the decomposition of the anomalous complexes at fixed momentum and level with respect to the ghost number can be summarized in the following form:

$$\mathcal{A}^N(p) = \begin{cases} \bigoplus_{r \geq -\frac{1}{2}} \mathcal{A}^{N,r}(p) & ; \quad \xi < 1 \\ \bigoplus_{r \geq -\frac{3}{2}} \mathcal{A}^{N,r}(p) & ; \quad \xi = 1 \quad . \end{cases} \quad (43)$$

It is worth to make a comment on the differences of the polarized and anomalous co-chains. From the conditions (33) and the identity (31) it follows that the polarized complex is generated by ghost modes c_{-m} , $m \geq 0$ and in addition b_{-1} anti-ghost in the degenerate case. The anomalous complex contains much more co-chains. Besides of the above, polarized elements, there are co-chains containing ghost/anti-ghost clusters. The clusters are generated by:

$$\begin{aligned} G_{-r,-s} &:= [J_\xi^{(-)}, (c_{-r} + c_{-s})] (c_{-r} + c_{-s}) - G_{-2r} - G_{-2s} \quad \text{and} \\ G_{-2m} &:= [J_\xi^{(-)}, c_{-m}] c_{-m}, \quad m, r \neq s > 0 \quad . \end{aligned} \quad (44)$$

From the definition (44) it immediately follows that these elements (of ghost number zero) are scalars with respect to $sl(2, \mathbb{C})$ -Lie algebra generated by the anomaly i.e. $[J_\xi^{(\pm)}, G_{(\cdot)}] = 0$.

As in the critical case there are several complexes associated with the anomalous BRST complex (32) and its polarized subcomplex (33). Although only the cohomologies of the polarized subcomplex will be calculated and identified, it is worth to define the associated complexes for the full anomalous complex. The corresponding polarized structures can be obtained by simple restriction to the subspace defined in (33).

First of all one introduces the on-mass-shell complex:

$$\mathcal{A}_0 := \ker (L_0^{\text{tot}}|_{\mathcal{A}}) = \bigoplus_{N \geq 0} \int_{S_N} d\mu^N(p) \mathcal{A}^N(p), \quad (45)$$

which contains the roots of all non-trivial cohomology classes of the anomalous complex (32). The elements of $\mathcal{A}^N(p)$ are supported on the mass-shells S_N defined by the equations:

$$p^2 = -2\alpha (N^{\text{str}} + N^{\text{gh}} + 2\beta - a) \quad , \quad N = N^{\text{str}} + N^{\text{gh}} \quad . \quad (46)$$

The property that any non-trivial cohomology class of (32) stems from the element of (45) follows from the fact that the operator $L_0^{\text{tot}} = \mathcal{L}_0 + L_0 - a$ is diagonal in the space (32) and moreover it is cohomologically trivial: $L_0^{\text{tot}} = \{b_0, D\}$. Consequently any D -closed element outside its kernel is exact (e.g. [7]). Hence the cohomologies of the on-mass-shell complex (\mathcal{A}_0, D) are identical with these of (32).

Once the complex is restricted to on-mass-shell cochains and it is observed that the ghost c_0 (corresponding to the kinetic operator L_0^{tot}) does not contribute to the mass spectrum it is natural to get rid of it and to introduce a relative complex:

$$\mathcal{A}_{\text{rel}}(p) := \{ \Psi \in \mathcal{A}_0(p) \ ; \ b_0 \Phi = 0 \} \ , \ \mathcal{A}_0(p) = \mathcal{A}_{\text{rel}}(p) \oplus c_0 \mathcal{A}_{\text{rel}}(p) \ , \quad (47)$$

with the differential:

$$D_{\text{rel}} := D - L_0^{\text{tot}} c_0 - M b_0 \ ; \ M = -2 \sum_{m>0} m c_{-m} c_m = \{D, c_0\} \ . \quad (48)$$

The square of relative differential can be calculated on the total differential space (24) and there it gives:

$$D_{\text{rel}}^2 = \frac{1}{12} ((d - 25) + 48\beta) 1 J_{\xi}^{(+)} - M L_0^{\text{tot}} \ . \quad (49)$$

The first term on the right hand side vanishes on anomalous cochains of \mathcal{A} while the second one is zero on the mass-shell. Hence D_{rel} is nilpotent on \mathcal{A}_{rel} .

Since the ghost zero mode c_0 is absent in \mathcal{A}_{rel} it is natural and convenient to introduce the integral grading in \mathcal{A}_{rel} by eigenvalues of relative ghost number operator:

$$\text{gh}_0 = \sum_{m>0} (b_{-m} c_m - c_{-m} b_m) \ . \quad (50)$$

This change of grading amounts to the shift $r \rightarrow r + \frac{1}{2}$ in the degrees (39) of all cochains from \mathcal{A}_{rel} .

The anomalous relative complex $\mathcal{A}_{\text{rel}}(p)$ admits (as in critical case [7]) richer grading structure than the one defined by the eigenvalues of the relative ghost number operator(50):

$$\mathcal{A}_{\text{rel}}^r(p) = \bigoplus_{a-b=r} \mathcal{A}_b^a \ , \quad (51)$$

where a, b denote the ghost c_{-m} excitation degree and anti-ghost b_{-m} excitation degree respectively.

The possibility to decompose any highest weight vector of $\mathcal{A}_{\text{rel}}^r(p)$ into bigraded components which are also of highest weights follows from the fact that the following operator:

$$\overline{\text{gh}}_0 = \sum_{m>0} (b_{-m} c_m + c_{-m} b_m) \ , \quad (52)$$

which counts the sums $a + b$, is central with respect to the $sl(2, \mathbb{C})$ algebra of (41).

The bigraded structure (51) makes it sensible and useful to decompose the relative differential accordingly $D_{\text{rel}} = \mathcal{D} + \overline{\mathcal{D}}$:

$$\begin{aligned} \mathcal{D} &= \sum_{m>0} L_{-m} \otimes c_m + \sum_{m>0} c_m \tau_{-m} + \partial \ , \\ \overline{\mathcal{D}} &= \sum_{m>0} L_m \otimes c_{-m} + \sum_{m>0} c_{-m} \tau_m + \overline{\partial} \ , \end{aligned} \quad (53)$$

where:

$$\partial = -\frac{1}{2} \sum_{m,k>0} (m-k) b_{-k-m} c_k c_m , \quad \bar{\partial} = -\frac{1}{2} \sum_{m,k>0} (m-k) c_{-m} c_{-k} b_{k+m} ,$$

and:

$$\tau_m = \sum_{k>m} (m+k) b_{m-k} c_k , \quad \tau_{-m} = \sum_{k>m} (m+k) c_{-k} b_{k-m} ; m > 0 .$$

It is remarkable³ property that independently of the presence of the anomaly the differentials \mathcal{D} and $\bar{\mathcal{D}}$ are nilpotent on the whole ghost extended string space (24). The anomaly is hidden in their anticommutator:

$$\begin{aligned} \mathcal{D}^2 &= 0 , \quad \bar{\mathcal{D}}^2 = 0 , \\ \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} &= \frac{1}{12} ((d-25) + 48\beta) 1 \otimes J_\xi^{(+)} - ML_0^{\text{tot}} , \end{aligned}$$

and both differentials obviously commute with $J_\xi^{(+)}$.

The above relations define the structure of $sl(1|1; \mathbb{R})$ Lie superalgebra. This structure was already used within slightly different, although related approach to quantum constrained systems in [9].

The bigraded structure⁴ of the anomalous (relative) complex can be most clearly summarized in the form of the following diagram:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow \mathcal{D} & & \downarrow \mathcal{D} & & \\ \dots & \xrightarrow{\bar{\mathcal{D}}} & \mathcal{A}_b^a & \xrightarrow{\bar{\mathcal{D}}} & \mathcal{A}_b^{a+1} & \xrightarrow{\bar{\mathcal{D}}} & \dots \\ & & \downarrow \mathcal{D} & & \downarrow \mathcal{D} & & \\ \dots & \xrightarrow{\bar{\mathcal{D}}} & \mathcal{A}_{b-1}^a & \xrightarrow{\bar{\mathcal{D}}} & \mathcal{A}_{b-1}^{a+1} & \xrightarrow{\bar{\mathcal{D}}} & \dots \\ & & \downarrow \mathcal{D} & & \downarrow \mathcal{D} & & \\ & & \vdots & & \vdots & & \end{array} \tag{54}$$

The form (53) of $\bar{\mathcal{D}}$ and \mathcal{D} differentials suggests that horizontal maps impose the constraints on the states, while the vertical maps implement the gauge transformations. The comment on the corresponding structure of the polarized complex must be added here. In the case of non-degenerate anomaly the only admissible anti-ghost degree of the polarized elements is zero i.e. $\mathcal{P}_b^a = 0$, $b > 0$. In the degenerate case the co-chains of anti-ghost degree equal to one (created by b_{-1} mode) are present and there is residual gauge symmetry generated by L_{-1}^{tot} operator.

This structure is most clearly visible in the form of the relative differential restricted to the polarized complex. Since $\partial|_{\mathcal{P}} \equiv 0$, one has on \mathcal{P} :

$$D_{\text{rel}} = \bar{\mathcal{D}} + \delta_{(\xi-1)}(L_{-1} + \tau_{-1})c_1 , \tag{55}$$

³and in fact it seems to be universal within the wide class of constrained systems

⁴This structure is in fact induced from the one of the total differential space [7].

where $\tau_{-1} = \sum_{k>0} (k+2)c_{-k-1}b_k$ is that of (53).

The bigraded structure of the polarized relative complex, as one will see in the next section, appears to be very useful in the computation of polarized cohomologies and in identification of their representatives in terms of physical states mentioned in the first chapter.

3 Cohomologies of the polarized BRST complex

This chapter is devoted to the computation of the cohomology spaces of polarized anomalous complexes introduced in the previous section. The identification of the representatives of non-zero cohomology classes with the elements of the physical space (10) of "old covariant" approach will be given explicitly too.

The cohomology spaces of the total complexes (33) may be formally reconstructed as a direct sums/integrals out of cohomology spaces:

$$H^r(p) := \frac{Z^r(p)}{B^r(p)} \ ; \ Z^r(p) := \ker D|_{\mathcal{P}^r(p)} \ , \ B^r(p) = \text{im } D|_{\mathcal{P}^{r-1}(p)} \ , \quad (56)$$

of the corresponding subcomplexes located at fixed momentum and level.

The convention to denote the spaces of cocycles by the root letter Z and those of coboundaries by the root letter B will be kept throughout this paper.

The relative cohomology spaces located at fixed momentum are defined in exactly analogous way. It is only worth to introduce the bigraded cohomology spaces corresponding to the structure illustrated in the diagram (54):

$$\overline{\mathcal{H}}_b^a(p) = \frac{\overline{\mathcal{Z}}_b^a(p)}{\overline{\mathcal{B}}_b^a(p)} \ , \ \mathcal{H}_b^a(p) = \frac{\mathcal{Z}_b^a(p)}{\mathcal{B}_b^a(p)} \ . \quad (57)$$

The spaces $\overline{\mathcal{H}}_b^a(p)$ and $\mathcal{H}_b^a(p)$ denote the cohomology spaces of $\overline{\mathcal{D}}$ and \mathcal{D} respectively. It should be mentioned that for non-degenerate anomaly the co-boundaries of \mathcal{D} are zero and all the elements of \mathcal{P}_0^a are \mathcal{D} co-cycles. Hence $\mathcal{H}_0^a(p) = \mathcal{P}_0^a(p)$ in this case.

The order of calculating the cohomologies of respective anomalous complexes will be exactly opposite to the one they were introduced in the previous section.

First of all it will be shown that an analog of Dolbeaut-Grothendieck vanishing theorem for bigraded cohomologies is true. Using this result it will be proved that also the relative classes do vanish for non zero (positive in anomalous case) ghost numbers. The transparent representation of the non trivial relative cohomology spaces will be given in terms of bigraded cocycles. This will in turn allow one to identify the relative classes with the physical states of "old covariant formalism" mentioned in the first chapter. Finally the absolute cohomologies will be reconstructed out of relative ones.

It is worth to mention that nothing like Poincare-Serre duality [7] can be used in the anomalous case. This is a consequence of the fact that there are no anomalous cochains of arbitrarily low negative ghost number and the natural pairing induced from full differential space \mathcal{C} of (24) is highly degenerate on the anomalous complex as well as on

its polarized subcomplex. Therefore one cannot use the canonical duality properties of the respective cohomology spaces. For this reason the cohomologies of $\overline{\mathcal{D}}$ and \mathcal{D} should be calculated independently. Due to the fact that the differential \mathcal{D} has residual form of (55) it is not necessary to know its cohomologies in order to prove the Vanishing Theorem for the relative classes of non-zero ghost number. It is also not needed for the identification of the representatives of non-vanishing classes with the physical states (10) of "old covariant approach".

As in [7], [17] and [18] the method of filtered complexes can be successfully used also here. For the sake of completeness the steps of reasoning will be repeated.

First, one has to choose the momentum supporting the space of cochains. Once a non-zero momentum p is fixed one may always find an adapted light cone basis⁵ $\{e_{\pm}, e_i\}_{i=1}^{d-2}$ of the momentum space such that $p^+ := e_+ \cdot p \neq 0$. The Virasoro operators of the string sector (8) written in the light-cone basis adapted to the momentum p take the form:

$$L_m = - \frac{1}{\sqrt{\alpha}} p^+ a_m^- + L_m^{\text{tr}} + L_m^{\text{Li}} - \sum_{n \neq 0} a_{n+m}^+ a_{-n}^- - \frac{1}{\sqrt{\alpha}} p^- a_m^+ . \quad (58)$$

The symbols L_m^{tr} and L_m^{Li} denote the operators given by standard expressions (8) in transverse and Liouville modes respectively.

Next, one can introduce an additional gradation in the spaces (33) by assigning the following filtration degrees to the elementary matter/ghost modes:

$$\begin{aligned} \deg(a_m^-) &= -1 , \quad \deg(b_m) = -1 \\ \deg(a_m^+) &= +1 , \quad \deg(c_m) = +1 \\ \deg(a_m^i) &= 0 , \quad \deg(u_m) = 0 ; \quad m \in \mathbb{Z} \setminus \{0\} , \quad 1 \leq i \leq d-2 . \end{aligned} \quad (59)$$

The spaces $\mathcal{P}_b^a(p)$ of bigraded complex (51) decompose into direct sums of filtration homogeneous components: $\mathcal{P}_b^a(p) = \bigoplus_f \mathcal{P}_{b,f}^a(p)$. According to this decomposition the differential $\overline{\mathcal{D}}$ decays into three parts:

$$\overline{\mathcal{D}} = \overline{\mathcal{D}}_{(0)} + \overline{\mathcal{D}}_{(1)} + \overline{\mathcal{D}}_{(2)} , \quad \overline{\mathcal{D}}_{(0)} = - \sum_{m>0} \frac{1}{\sqrt{\alpha}} p^+ a_m^- c_{-m} . \quad (60)$$

The operators $\overline{\mathcal{D}}_{(i)}$, $i = 1, 2$ can be easily read off from (53) and (58) but their explicit form is not needed. It is only worth to mention that they both act by rising the filtration degree:

$$\overline{\mathcal{D}}_{(i)} : \mathcal{P}_{b,f}^a(p) \rightarrow \mathcal{P}_{b,f+i}^{a+1}(p) . \quad (61)$$

Out of (60) only the component $\overline{\mathcal{D}}_{(0)}$ of filtration degree zero will be in explicit use. It is nilpotent and defines the cohomology space $\overline{\mathcal{H}}_{b,f}^a(p)$ localized at fixed filtration degree f .

Equipped with the above tools, by a slight modification of the arguments used in [7], one may prove the following important

⁵Consisting of two light like vectors $e_{\pm}^2 = 0$, $e_+ \cdot e_- = -1$ and an orthonormal basis $\{e_i\}_{i=1}^{d-2}$ of Euclidean transverse space.

Lemma 3.1

$$\overline{\mathcal{H}}_{b,f}^a(p) = 0 ; \quad a > 0 , \quad p \neq 0 .$$

Proof: The operator of filtration degree zero:

$$\overline{\mathcal{R}} = \sum_{m>0} (c_{-m}b_m - \frac{1}{m}a_{-m}^+a_m^-) ,$$

counts the total degree of c_m ghost and a_m^+ string excitations hence it is non-negative. From the statement (34) it follows that it acts within the spaces \mathcal{P}_b^a of polarized cochains contained in (51). The operator $\overline{\mathcal{R}}$ is exact:

$$\overline{\mathcal{R}} = \{ \overline{\mathcal{D}}_{(0)} , \overline{\mathcal{K}} \} , \quad \text{where} \quad \overline{\mathcal{K}} = \frac{\sqrt{\alpha}}{p^+} \sum_{m>0} \frac{1}{m} a_{-m}^+ b_m , \quad (62)$$

on the whole representation space \mathcal{C}_b^\bullet and again due to (34) also on \mathcal{P}_b^\bullet .

Let Ψ be $\overline{\mathcal{D}}_{(0)}$ closed cochain from $\mathcal{P}_b^\bullet(p)$. It may be assumed that Ψ is an eigenstate of $\overline{\mathcal{R}}$: $\overline{\mathcal{R}}\Psi = s\Psi$. From the relation (62) it follows that

$$\Psi = \frac{1}{s} \overline{\mathcal{D}}_{(0)} \overline{\mathcal{K}} \Psi , \quad (63)$$

for any cocycle of $\overline{\mathcal{R}}$ -degree $s \neq 0$. Consequently all $\overline{\mathcal{D}}_{(0)}$ closed states not in the kernel of $\overline{\mathcal{R}}$ (in particular those with $a > 0$) are exact. \square

Making use of the above statement it is possible to prove the strict counterpart of the Dolbeaut-Grotendieck lemma of classical complex geometry [15] on vanishing of bigraded cohomologies (57) of $\overline{\mathcal{D}}$.

Lemma 3.2 (Dolbeaut - Grotendieck)

$$\overline{\mathcal{H}}_b^a(p) = 0 ; \quad a > 0 , \quad p \neq 0 .$$

Proof: Since the filtration degree is bounded at any fixed level the cochain of bidegree (a, b) can be decomposed into finite sum of homogeneous components with respect to the filtration degree: $\Psi_b^a = \sum_{i \geq m} \Psi_{b,i}^a$. The equation $\overline{\mathcal{D}}\Psi_b^a = 0$ written in terms of (60) implies a chain of equations for homogeneous constituents and in particular $\overline{\mathcal{D}}_{(0)}\Psi_{b,m}^a = 0$ for the component of lowest filtration degree. Since the cohomologies of $\overline{\mathcal{D}}_{(0)}$ are trivial $\Psi_{b,m}^a = \overline{\mathcal{D}}_{(0)}F_{b,m}^{a-1}$ (Lemma 3.1). The lowest filtration component of an equivalent element $\Psi_b'^a = \Psi_b^a - \overline{\mathcal{D}}F_{b,m}^{a-1}$ is of degree at least $m+1$. The procedure repeated appropriately many times leads to the conclusion that $\Psi_b^a = \overline{\mathcal{D}}\Phi_b^{a-1}$ for some Φ_b^{a-1} . \square

The above lemma implies directly the Vanishing Theorem for relative cohomology and provides a convenient description of non-vanishing classes at ghost number zero in terms of bigraded cocycles.

Theorem 3.1 (Vanishing Theorem)

$$1) \quad H_{\text{rel}}^r(p) = 0 ; \quad r \neq 0 , \quad p \neq 0 \quad (\text{on-shell})$$

$$2) \quad H_{\text{rel}}^0(p) \sim \begin{cases} \bar{\mathcal{Z}}_0^0(p) & ; \quad \xi < 1 \\ \bar{\mathcal{Z}}_0^0(p) / \mathcal{D}(b_{-1}\bar{\mathcal{Z}}_0^0(p)) & ; \quad \xi = 1 . \end{cases}$$

Proof: 1) Take $\Psi^r \in Z_{\text{rel}}^r(p)$. In the case of $r > 0$ Ψ^r has the following bidegree decomposition: $\Psi^r = \Phi_0^r + \delta_{(\xi-1)}\Phi_1^{r+1}$. The equation for Ψ^r to be a relative co-cycle reads:

$$\delta_{(\xi-1)}\bar{\mathcal{D}}\Phi_1^{r+1} + \bar{\mathcal{D}}\Phi_0^r + \delta_{(\xi-1)}\mathcal{D}\Phi_1^{r+1} = 0 .$$

In particular it should be $\bar{\mathcal{D}}\Phi_1^{r+1} = 0$. Taking into account Dolbeaut - Grotendieck vanishing lemma for $\bar{\mathcal{H}}_b^a(p)$ one has the solution $\Phi_1^{r+1} = \bar{\mathcal{D}}F_1^r$ for some co-chain F_1^r of lower degree. Taking an equivalent element $\Psi^r \sim \tilde{\Psi}^r = \Psi^r - \delta_{(\xi-1)}D_{\text{rel}}F_1^r = \tilde{\Phi}_0^r$ one gets $\bar{\mathcal{D}}\tilde{\Phi}_0^r = 0$. Hence again due to Dolbeaut-Grotendieck vanishing lemma $\tilde{\Phi}_0^r = \bar{\mathcal{D}}\tilde{F}_0^{r-1} = D_{\text{rel}}\tilde{F}_0^{r-1}$. Finally $\Psi^r \sim D_{\text{rel}}\tilde{F}_0^{r-1}$ and $H_{\text{rel}}^r(p) = 0$ for $r > 0$.

For the negative ghost number (only $r = -1$ has to be taken into account and only in the case of $\xi = 1$) one should take $\Psi^{-1} = b_{-1}\Phi_0^0$. The condition for Ψ^{-1} to be a co-cycle with respect to D_{rel} amounts to $\bar{\mathcal{D}}\Phi_0^0 = 0$ and $\{\mathcal{D}, b_{-1}\}\Phi_0^0 = 0$. The first equation follows from the fact that $\bar{\mathcal{D}}$ anti-commutes with the first anti-ghost mode b_{-1} (which is not in fact true for the higher ones) and means that Φ_0^0 is (up to ghost vacuum factor) the physical state in the sense of (10) i.e. $L_n\Phi_0^0 = 0$ for $n > 0$. On the other hand from (55) it follows that the second equation reduces to $L_{-1}\Phi_0^0 = 0$. Hence also $L_1L_{-1}\Phi_0^0 = [L_1, L_{-1}]\Phi_0^0 = 0$ and as a consequence of the mass-shell condition (10) and vanishing of the anomaly at the first ghost/antighost mode one obtains $L_0\Phi_0^0 = \Phi_0^0 = 0$. 2) Taking $\Psi^0 = \Phi_0^0 + \delta_{(\xi-1)}b_{-1}\Phi_0^1$ of the most general bi-degree decomposition in the polarized complex \mathcal{P} and imposing $D_{\text{rel}}\Psi^0 = 0$, one obtains the following conditions for its homogeneous components:

$$\bar{\mathcal{D}}\Phi_0^1 = 0 \quad \text{and} \quad \bar{\mathcal{D}}\Phi_0^0 + \delta_{(\xi-1)}\mathcal{D}(b_{-1}\Phi_0^1) = 0 .$$

Again from Dolbeaut-Grotendieck vanishing lemma one obtains $\Phi_0^1 = \bar{\mathcal{D}}F_0^0$ for some F_0^0 from \mathcal{P}_0^0 . Taking an equivalent element $\Psi^0 \sim \tilde{\Psi}^0 = \Psi^0 + \delta_{(\xi-1)}D_{\text{rel}}(b_{-1}F_0^0) = \tilde{\Phi}_0^0$ one obtains $\mathcal{D}\tilde{\Phi}_0^0 = 0$. Hence any non trivial class from $H_{\text{rel}}^0(p)$ stems from the co-cycle of $\bar{\mathcal{Z}}_0^0(p)$. Further identification of $\bar{\mathcal{Z}}_0^0(p)$ cocycles amounts to $\Psi_0^0 - \tilde{\Psi}_0^0 = \delta_{(\xi-1)}\mathcal{D}(b_{-1}F_0^0)$ with F_0^0 being the physical state i.e. $\bar{\mathcal{D}}F_0^0 = 0$. \square

The result above and the statement 2) in particular, allows one to identify the representatives of non vanishing cohomology classes with the physical states (10) of "old covariant quantization". It is enough to notice (as it was already done in the proof of the above Theorem) that the condition for $\Psi_0^0(p) = v(p) \otimes \omega$; $v(p) \in \mathcal{H}(p)$ to be a cocycle reads:

$$0 = \bar{\mathcal{D}}\Psi_0^0(p) = \sum_{m>0} L_m v(p) \otimes c_{-m}\omega ,$$

and consequently:

$$\bar{\mathcal{Z}}_0^0(p) = \mathcal{H}_{\text{phys}}(p) \otimes \omega , \tag{64}$$

where $\mathcal{H}_{\text{phys}}(p)$ is the subspace (10) of the original covariant space (6) of string states. In non-degenerate case ($\xi < 1$) there is no any additional equivalence (gauge) relation between them. For $\xi = 1$ one obtains residual gauge symmetry generated by L_{-1} constraint operator. This appearance is tightly related to the fact that the Gupta-Bleuler subalgebra, used to define the physical states (10), supplemented by L_{-1} , is first of all anomaly free (as the corresponding constraints are of first class on the space of the total complex (24)) and in addition, it forms the parabolic subalgebra of (9) in the sense of [16].

The above situation should be compared and contrasted with the one encountered in the critical massive string model. As it was demonstrated in [7] the relative cohomology space had the form of the quotient:

$$H_{\text{rel}}^0(p) \sim \overline{\mathcal{Z}}_0^0(p) / \mathcal{D}\overline{\mathcal{Z}}_1^0(p) ,$$

where the space $\mathcal{D}\overline{\mathcal{Z}}_1^0(p)$ of "gauge states" was generated by all $\{L_{-m}^{\text{tot}}; m > 0\}$ operators and was identified with the radical of $\mathcal{H}_{\text{phys}}(p)$ containing all null vectors. There are also null states in the anomalous case (those of discrete series (12)), but according to Vanishing Theorem above ($\xi < 1$) they cannot be gauged away in the sense of cohomology.

The remaining part of this section will be devoted to the description of the total (absolute) cohomologies of the polarized complex. In order to reconstruct the absolute cohomology spaces out of relative ones, one should follow the way presented in [7] and [19]. The general reasoning (which enables one to treat the degenerate and non-degenerate cases simultaneously) will be reported here for completeness.

First of all, it will be demonstrated that due to Vanishing Theorem for relative cohomologies, all the absolute classes from $H^s(p)$ are zero for $s \neq \pm\frac{1}{2}$. According to the decomposition of (47) any (on-mass-shell with $p \neq 0$) element of absolute degree $s = r + \frac{1}{2}$ ($r \in \mathbb{Z}$) can be written as:

$$\Psi^{r+\frac{1}{2}} = c_0\Phi^r + \Phi^{r+1} \quad \text{with} \quad \Phi^r, \Phi^{r+1} \in \mathcal{P}_{\text{rel}}(p) . \quad (65)$$

Assuming that $\Psi^{r+\frac{1}{2}}$ is closed: $D\Psi^{r+\frac{1}{2}} = 0$, one obtains the following equations for relative components:

$$D_{\text{rel}}\Phi^r = 0 \quad \text{and} \quad D_{\text{rel}}\Phi^{r+1} + M\Phi^r , \quad (66)$$

where $M = \{D, c_0\}$ is that of (48).

If $r \neq 0$, then according to Vanishing Theorem, the first equation is solved by $\Phi^r = D_{\text{rel}}\chi^{r-1}$ for some χ^{r-1} from $\mathcal{P}_{\text{rel}}^{r-1}(p)$. Inserting this solution into second equation one obtains $D_{\text{rel}}(\Phi^{r+1} + M\chi^{r-1}) = 0$.

For $r \neq -1$, again due to Vanishing Theorem, one gets the solution $\Phi^{r+1} + M\chi^{r-1} = D_{\text{rel}}\xi^r$ for some ξ^r in $\mathcal{P}_{\text{rel}}^r(p)$.

Making the gauge shift:

$$\Psi^{r+\frac{1}{2}} \rightarrow \hat{\Psi}^{r+\frac{1}{2}} = \Psi^{r+\frac{1}{2}} + D(c_0\chi^{r-1}) = D\xi^r , \quad (67)$$

one obtains the element which is cohomologous to zero.

Hence all absolute cohomology spaces except of possibly $H^{\pm\frac{1}{2}}(p)$ are zero.

In the second step it will be shown that the absolute classes $H^{\pm\frac{1}{2}}(p)$ can be reconstructed out of these of $H_{\text{rel}}^0(p)$.

In order to get $H^{-\frac{1}{2}}(p)$ one introduces the natural inclusion map on relative cocycles:

$$Z_{\text{rel}}^0(p) \ni \Psi^0 \rightarrow i_-(\Psi^0) := \Psi^0 \in Z^{-\frac{1}{2}}(p) . \quad (68)$$

Since $Di_- = i_-D_{\text{rel}}$, the map (68) is well defined and transforms relatively exact elements into absolutely exact ones. Hence it determines the unique mapping:

$$H_{\text{rel}}^0(p) \ni [\Psi^0]_{\text{rel}} \rightarrow i_-^*[\Psi^0]_{\text{rel}} = [i_-(\Psi^0)]_{\text{abs}} \in H^{-\frac{1}{2}}(p) , \quad (69)$$

of the respective cohomology spaces.

The situation is a bit more complicated with $H^{\frac{1}{2}}(p)$. It is clear that the corresponding mapping should be generated by multiplication of relative cocycles by the ghost zero mode c_0 . It is however not true that simple multiplication transforms relative cocycles into absolute ones. One has instead:

$$D(c_0\Psi^r) = -c_0D_{\text{rel}}\Psi^r + M\Psi^r . \quad (70)$$

Nevertheless the multiplication map can be corrected in appropriate way.

For this to be done one should observe first, that $M\Psi^r$ is always D -exact for Ψ^r from $Z_{\text{rel}}^r(p)$. For $r \neq 0$ and Ψ^r in $Z_{\text{rel}}^r(p)$, according to Vanishing Theorem, one has $\Psi^r = D_{\text{rel}}\chi^{r-1}$ and $M\Psi^r = D(M\chi^{r-1})$. For $r = 0$ the closed element $M\Psi^0$ is of degree 2. Hence, again due to Vanishing Theorem, it is cohomologically trivial: $M\Psi^0 = D_{\text{rel}}\chi^1 = D\chi^1$ for some χ^1 in $\mathcal{P}_{\text{rel}}^1(p)$.

Consequently, one may introduce the mapping:

$$Z_{\text{rel}}^r(p) \ni \Psi^r \rightarrow P(M\Psi^r) \in \mathcal{P}_{\text{rel}}^{r+1}(p) , \quad (71)$$

which associates the primary of $M\Psi^r$ with any Ψ^r from Z_{rel}^r . This mapping is not uniquely determined but it can always be chosen in such a way that $P(MD_{\text{rel}}\Psi^r) = M\Psi^r$.

Using (71) one may introduce the corrected multiplication map:

$$Z_{\text{rel}}^0(p) \ni \Psi^0 \rightarrow c(\Psi^0) := c_0\Psi^0 - P(M\Psi^0) \in Z^{\frac{1}{2}}(p) . \quad (72)$$

It transforms relative cocycles into absolute ones and moreover, due to the appropriate choice of primaries, it satisfies:

$$c(D_{\text{rel}}\chi^r) = -D(c_0\chi^r) . \quad (73)$$

Hence one obtains well defined map of cohomology spaces:

$$H_{\text{rel}}^0(p) \ni [\Psi^0]_{\text{rel}} \rightarrow i_+^*[\Psi^0]_{\text{rel}} = [c(\Psi^0)]_{\text{abs}} \in H^{\frac{1}{2}}(p) . \quad (74)$$

One is now in a position to prove the following

Theorem 3.2 (Absolute Cohomologies)

The mappings:

$$H^{-\frac{1}{2}}(p) \xleftarrow{i_-^*} H_{\text{rel}}^0(p) \xrightarrow{i_+^*} H^{\frac{1}{2}}(p) , \quad p \neq 0 ,$$

are the isomorphisms of the cohomology spaces.

Proof: Any cocycle $\Psi^{-\frac{1}{2}} \in Z^{-\frac{1}{2}}(p)$ is of the form $\Psi^{-\frac{1}{2}} = c_0\Phi^{-1} + \Phi^0$ with $D_{\text{rel}}\Phi^{-1} = 0$ and $D_{\text{rel}}\Phi^0 + M\Phi^{-1} = 0$. The Vanishing Theorem and the absence of anomalous cochains of degree lower than -1 implies that $\Phi^{-1} = 0$ is the unique solution. Hence $\Psi^{-\frac{1}{2}} = \Phi^0$ and i_- is onto. In order to check injectivity one should notice that $\Psi^{-\frac{1}{2}} \sim \hat{\Psi}^{-\frac{1}{2}}$ in the sense absolute cohomology amounts to $\Phi^0 - \hat{\Phi}^0 = D_{\text{rel}}\chi^{-1}$ i.e. the relative equivalence. Hence i_-^* is an isomorphism.

Assuming that $\Psi^{\frac{1}{2}} = c_0\Phi^0 + \Phi^1$ is a cocycle one obtains $D_{\text{rel}}\Phi^0 = 0$ and $D_{\text{rel}}\Phi^1 + M\Phi^0 = 0$. From the first equation it follows that $M\Phi^0$ is closed and due to the considerations above one can take its primary to solve for Φ^1 : $\Phi^1 = -P(M\Phi^0)$. Hence $\Psi^{\frac{1}{2}} = c_0\Phi^0 - P(M\Phi^0) = c(\Phi^0)$. Consequently the map i_+^* is onto. Assume that $c(\Phi^0) \sim c(\hat{\Phi}^0)$ in the absolute space. This amounts to $c_0(\Phi^0 - \hat{\Phi}^0) - P(M\Phi^0) + P(M\hat{\Phi}^0) = -c_0D_{\text{rel}}\chi^0 + M\chi^0 + D_{\text{rel}}\chi^1$. Hence $\Phi^0 - \hat{\Phi}^0 = -D_{\text{rel}}\chi^0$ and $\chi^1 = -P(M\chi^0)$. This proves injectivity of i_+^* . \square

The general results obtained above together with 2) of Vanishing Theorem for relative cohomologies give the tractable representation of the absolute cohomology classes. In the non-degenerate case one has⁶

$$H^{-\frac{1}{2}}(p) \simeq \mathcal{H}_{\text{phys}}(p) \quad \text{and} \quad H^{\frac{1}{2}}(p) \simeq c_0\mathcal{H}_{\text{phys}}(p) ,$$

while for degenerate anomaly one obtains the quotients:

$$H^{-\frac{1}{2}}(p) \simeq \mathcal{H}_{\text{phys}}(p) / \sim \quad \text{and} \quad H^{\frac{1}{2}}(p) \simeq c_0\mathcal{H}_{\text{phys}}(p) / \sim .$$

The equivalence relation above is defined by: $\Psi \sim \Psi'$ if and only if $\Psi - \Psi' = L_{-1}\Phi$ for some Φ from $\mathcal{H}_{\text{phys}}$.

All the results of this chapter were obtained under essential assumption that the momentum underlying the polarized anomalous complex was non-zero.

There is a series of solutions of the mass-shell equation (46) admitting $p = 0$. They are all located in the discrete series (12) of unitary models. The numerical calculations show that there are perfect vacuum states in several spacetime dimensions at level $N = 0$, and zero momentum states at first excited level $N = 1$ in $d = 25$. For example, in $d = 4$, the perfect vacuum states exist for $m = 24$, $r = s = 20$ and $m = 242$, $r = s = 198$. The zero momentum states at level $N = 1$ are admissible for $d = 25$ and $m = 24$, $r = s = 10$ or $m = 242$, $r = s = 99$. The numerical calculations performed up to $m = 900$ indicate that one finds less and less solutions with m growing.

The absolute cohomologies for $p = 0$ case can be calculated directly. One finds immediately:

$$H^{-\frac{1}{2}}(0) = \mathbb{C}\Omega(0) \quad , \quad H^{\frac{1}{2}}(0) = \mathbb{C}c_0\Omega(0) \quad \text{for } N = 0 \quad , \quad (75)$$

and

$$H^{-\frac{1}{2}}(0) = \mathcal{H}_{\text{phys}}^1(0), \quad H^{\frac{1}{2}}(0) = c_0\mathcal{H}_{\text{phys}}^1(0) \oplus \mathbb{C}c_{-1}\Omega(0), \quad H^{\frac{3}{2}}(0) = \mathbb{C}c_0c_{-1}\Omega(0) \quad ,$$

⁶By slight abuse of notation, the space $\mathcal{H}_{\text{phys}}$ of (10) is identified with its image in the complex $\mathcal{H}_{\text{phys}} \otimes \omega$, where ω denotes the ghost vacuum.

for $N = 1$, where the space $\mathcal{H}_{\text{phys}}^1(0)$ above is generated by level one matter modes a_{-1}^μ , u_{-1} ; $\mu = 0, \dots, d-1$ out of the perfect matter/ghost vacuum state $\Omega(0)$.

One should notice the lack of duality at $N = 1$. The state $c_{-1}\Omega(0)$ cannot be eliminated because its primary $c_0b_{-1}\Omega(0)$ does not belong to polarized anomalous complex. For the same reason the dual partner $b_{-1}\Omega(0)$ of $H^{\frac{3}{2}}(0)$ is missing in the kernel of $J_\xi^{(+)}$.

The above considerations exhaust and finish the analysis of polarized anomalous cohomologies for non-critical massive string theories.

4 Final remarks

The cohomologies of the polarized anomalous complex for non-critical massive strings models were investigated in this paper. The proposed approach, although restricted to string models, seems to be an universal tool to describe the constrained systems with anomalies and possibly to develop a theory of their interactions.

The results of this paper indicate that the polarized anomalous complexes introduced here should give the resolutions of the spaces of physical states for the wide class of models with anomalies. It has to be stressed that the presented method does not distinguish the cases of whether the anomalies are of classical or quantum origin or both together.

The authors did not calculate the cohomologies of the full anomalous complex. This problem is left open. It would be interesting to determine them and to find their geometrical meaning and physical interpretation.

There is also an interesting problem related with the non-degenerate pairing on the anomalous complex and its polarized subcomplex. It appears that it can be introduced by the use of the anomaly itself. The anomaly defines the pseudo-Kaehler structure on \mathcal{A} and \mathcal{P} . The corresponding Kaehler-Laplace operators can be used to define the harmonic representatives of the non-vanishing classes. The work on this subject is advanced and the results will be presented in the forthcoming paper.

The authors intend to investigate the properties of the proposed approach in much wider context. The work on finite dimensional systems with anomalies (of the classical origin obviously) is in progress. The authors aim to investigate the field-theoretical models (with anomalies of both kinds - quantum and classical) in near future.

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